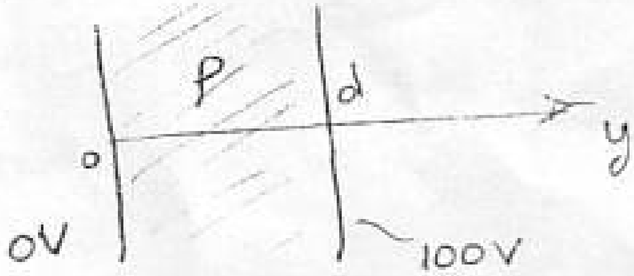
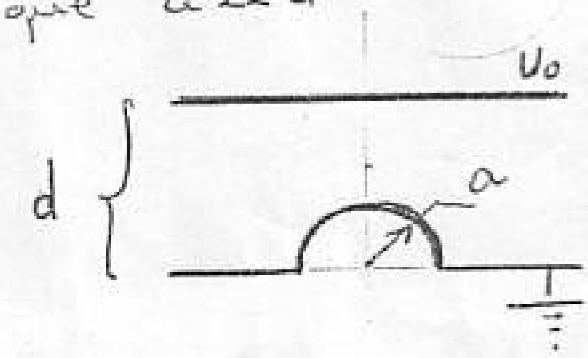
2da Práctica Calificada

(1) Si $\rho = \rho_0 \frac{y}{d}$, hallar el potencial electrostático y campo eléctrico para la región de la figura



(2) Um condensador de placas planas y paralelas, separadas por uma distancia d, una de las placas tiene uma pequeña protuberancia hemisférica de radio a. Se pide calcular el potencial electrostático y el campo eléctrico para $\theta = 0°$ y $r = d/2$. Si se sabe además que $a \ll d$

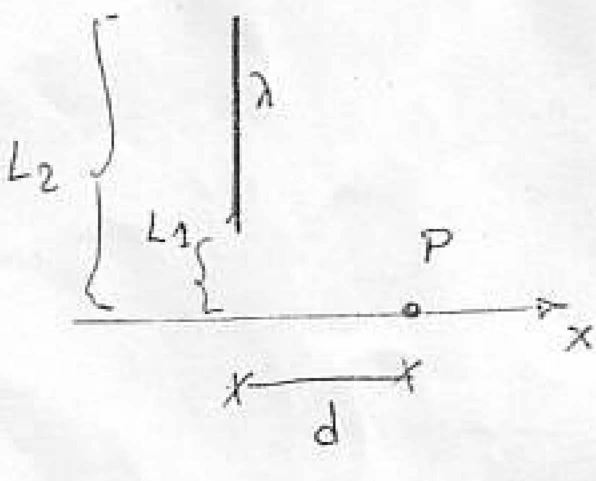


(3) Hallar el campo eléctrico em cualquier parte de las sgtes distribuciones
a) volumétricas de carga con simetría cilíndrica.

a) $\rho(r) = \rho_0 e^{-\frac{r}{a}}$ $\qquad 0 < r < \infty$

b) $\rho(r) = \rho_0 \frac{r}{a}$ $\qquad 0 < r < a$

(4) Hallar el potencial y el campo eléctrico en el punto P



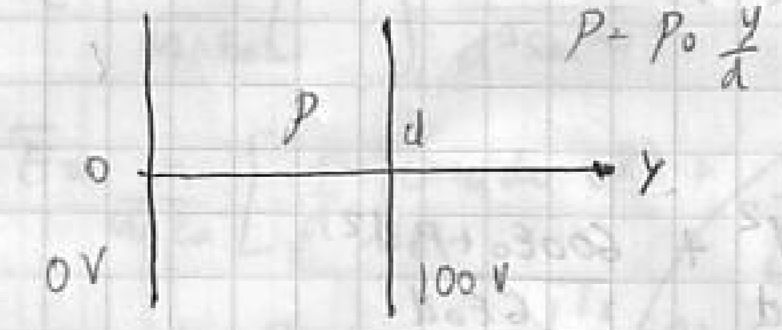
Examen de Teoría de campos Electromagnéticos

09

Alumno: Lope Choppa Gonzalo OIL

Código: 060047D 12-02-09

1) $\rho = \rho_0 \frac{y}{d}$



Solución: Usando condiciones de frontera en unidimensionales en coordenadas cartesianas: $\nabla^2 U = \cancel{\frac{\partial^2 U}{\partial x^2}} + \frac{\partial^2 U}{\partial y^2} + \cancel{\frac{\partial^2 U}{\partial z^2}} = -\frac{\rho_c}{\varepsilon_0}$ Ecuación de Laplace

~~$U(y) = Ay + B$ ...(1) Solución.~~ $\nabla^2 U = -\frac{\rho_c}{\varepsilon_0}$ ..(1) ρ_c = densidad de carga en el interior.

~~C.F: $y=0$; $U=0V$ Reemplazando en(1)~~

~~$y=d$; $U=100V$~~

~~$0 = A(0)+B$~~
~~$100 = Ad + B$~~ ~~$\Rightarrow 100 = Ad$~~

~~$\boxed{A = \frac{100}{d}}$ ... (2)~~

Existe Carga dentro del condensador: donde: $\rho_c = \frac{\rho_0 y}{d}$ Reemplazando en(1):

$$\Rightarrow \frac{\partial^2 U}{\partial y^2} = -\frac{\rho_0 \frac{y}{d}}{\varepsilon_0} = -\frac{\rho_0 y}{d\varepsilon_0}$$

5

Resolviendo: $\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right) = -\frac{\rho_0 y}{d\varepsilon_0} \Rightarrow \frac{\partial U}{\partial y} = -\frac{\rho_0 y^2}{2d\varepsilon_0} + A \Rightarrow U(y) = -\frac{\rho_0 y^3}{6d\varepsilon_0} + Ay + B$

$\therefore U(y) = -\frac{\rho_0 y^3}{6d\varepsilon_0} + Ay + B$ ....(2) Solución general.

CF: $\begin{cases} y=0 \ ; \ U=0V \\ y=d \ ; \ U=100V \end{cases}$ Reemplazando en(2):

$0 = -\frac{\rho_0 (0)^3}{6d\varepsilon_0} + A(0) + B$
$100 = -\frac{\rho_0 d^3}{6d\varepsilon_0} + Ad + B$ $\Big\} \Rightarrow 100 = -\frac{\rho_0 d^2}{6\varepsilon_0} + Ad$

Despejando a "A": $A = \frac{100}{d} + \frac{\rho_0 d^2}{6\varepsilon_0 d} = \frac{600\varepsilon_0 + \rho_0 d^2}{6\varepsilon_0 d}$ ... (3)

encontrando a "B": $B = 0$

Reemplazando en la solución general (2):

$$U(y) = -\frac{\rho_0 y^3}{6d\varepsilon_0} + Ay + B$$

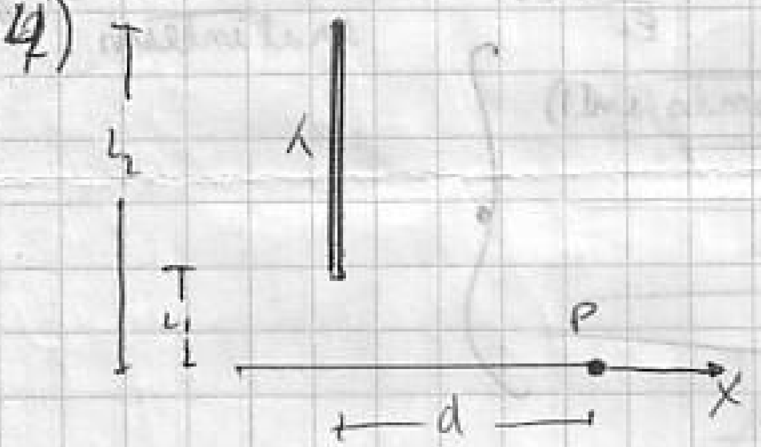
$$\therefore \quad U(y) = -\frac{\rho_0 y^3}{6d\varepsilon_0} + \left(\frac{600\varepsilon_0 + \rho_0 d^2}{6\varepsilon_0 d}\right) y$$

Hallando el campo eléctrico:

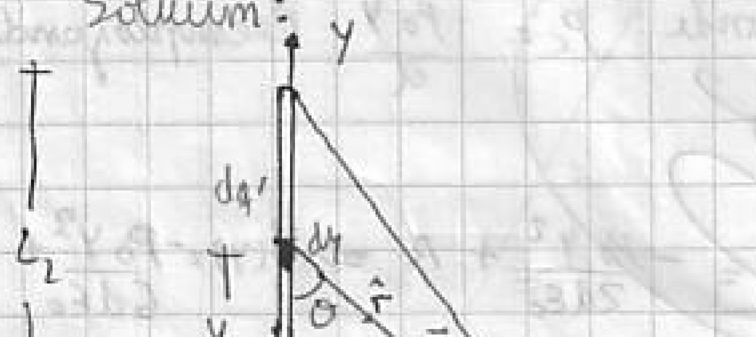
$$\bar{E}_{(y)} = -\bar{\nabla} U = -\frac{\partial U}{\partial y} = -\frac{\rho_0 y^2}{2\varepsilon_0 d} + \frac{600\varepsilon_0 + \rho_0 d^2}{6\varepsilon_0 d}$$

$$\therefore \quad E_{(y)} = -\frac{\rho_0 y^2}{2\varepsilon_0 d} + \frac{600\varepsilon_0 + \rho_0 d^2}{6\varepsilon_0 d}$$

4)



Solución:



$$dq' = \lambda\, dy \quad \ldots (1)$$

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq'}{|\bar{r}|^2}\,\hat{r} \quad \ldots (2)$$

donde: $\hat{r} = \dfrac{\bar{r}}{|\bar{r}|} \quad \ldots (3)$

$\bar{r} = y(-\hat{j}) + d(\hat{i})$ (componentes de $\bar{r}$)

$|\bar{r}| = \sqrt{d^2 + y^2}$

Reemplazando en (3): $\hat{r} = \dfrac{d(\hat{i}) + y(-\hat{j})}{\sqrt{d^2+y^2}} \quad \ldots (4)$

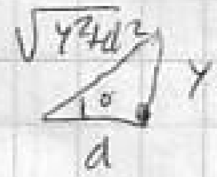
Reemplazando (1), (4) en (2):

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, dy}{(y^2+d^2)} \cdot \left(\frac{d\hat{i} + y(-\hat{j})}{\sqrt{y^2+d^2}}\right) =$$

$$\bar{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[d\int \frac{dy}{(y^2+d^2)^{3/2}}\,\hat{i} + \int \frac{y\, dy}{(y^2+d^2)^{3/2}}\,(-\hat{j}) \right]$$

haciendo: $tg\theta = \frac{y}{d}$ para los dos componentes

$dy = d\sec^2\theta\, d\theta$ Variación: $\{L_1 \le y \le L_2\}$



$\sin\theta = \frac{y}{\sqrt{y^2+d^2}}$

$\cos\theta = \frac{d}{\sqrt{y^2+d^2}}$

$$\Rightarrow \bar{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[d\int_{L_1}^{L_2} \frac{d\sec^2\theta\, d\theta}{d^3\sec^3\theta}(\hat{\imath}) + \int_{L_1}^{L_2} \frac{d\, tg\theta\,(d\sec^2\theta\, d\theta)}{d^3\sec^3\theta}(-\hat{\jmath})\right]$$

$$\bar{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{d}\int_{L_1}^{L_2}\cos\theta\, d\theta\,(\hat{\imath}) + \frac{1}{d}\int_{L_1}^{L_2}\frac{tg\theta}{\sec\theta}\, d\theta\,(-\hat{\jmath})\right]$$

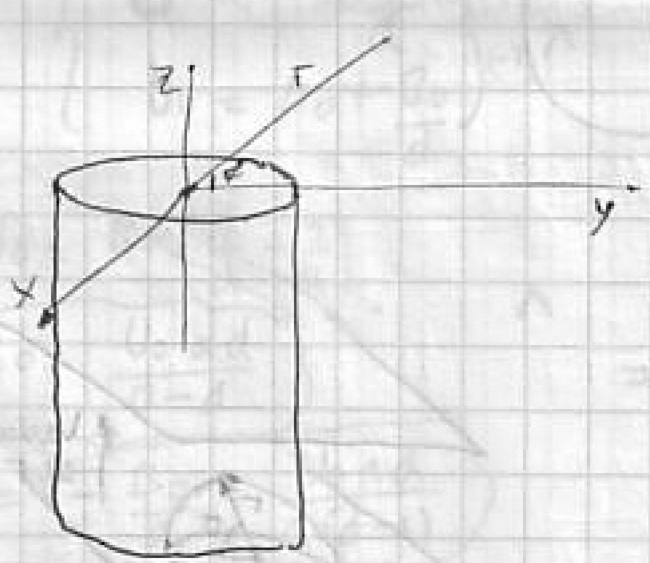
$$\bar{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{d}\sin\theta\Big|_{L_1}^{L_2}(\hat{\imath}) + \frac{1}{d}(-\cos\theta)\Big|_{L_1}^{L_2}(-\hat{\jmath})\right]$$

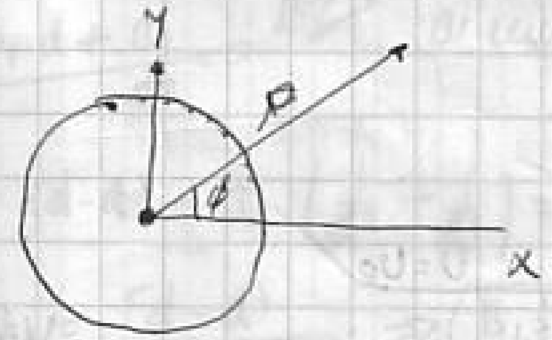
$$\bar{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{d}\left(\frac{y}{(y^2+d^2)^{1/2}}\right)\Big|_{L_1}^{L_2}(\hat{\imath}) + \frac{1}{d}\left(\frac{d}{\sqrt{y^2+d^2}}\right)\Big|_{L_1}^{L_2}(-\hat{\jmath})\right]$$

$$\bar{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{L_2}{d(L_2^2+d^2)^{1/2}} - \frac{L_1}{d(L_1^2+d^2)^{1/2}}\right]\hat{\imath} + \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{L_2^2+d^2}} - \frac{1}{\sqrt{L_1^2+d^2}}\right]\hat{\jmath}$$

3)

$0<r<\infty$



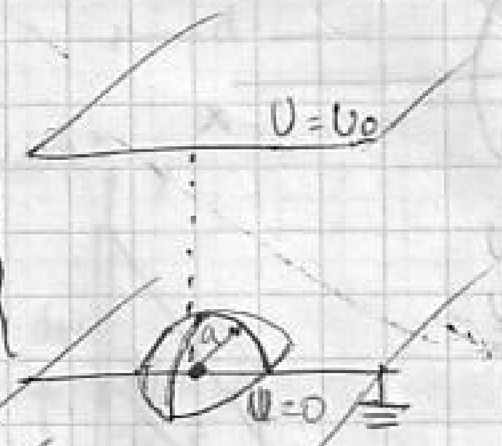


de: $\bar{E} = -\nabla U$

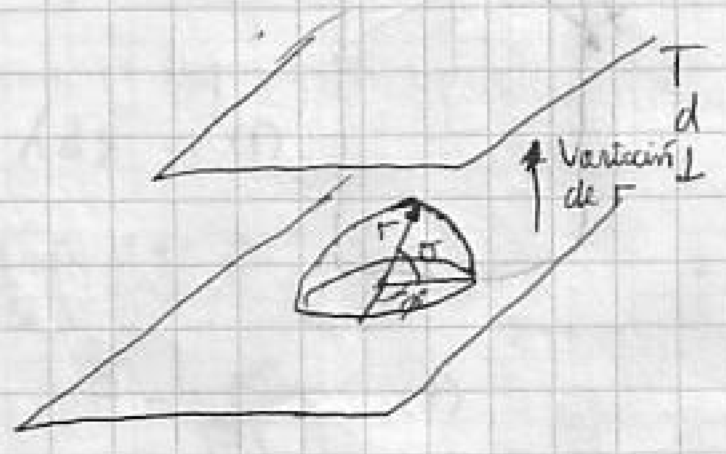
$\bar{E} = -\frac{\partial U}{\partial x}$

$U = -E$

2)



Vista:



Solución:

Como: $a << d$ entonces

$U(r=a)=0$

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial}{\partial \theta}\left(\operatorname{sen}\theta\frac{\partial U}{\partial \theta}\right) + \cancel{\frac{1}{r^2\operatorname{sen}^2\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial U}{\partial \phi}\right)} = 0$$

Como depende de r y θ ¡no depende ϕ:

$$\Rightarrow U_{(r,\theta)} = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right)\left[P_n \cos\theta\right] \quad \ldots (1)$$

donde:

$$C.F = \begin{cases} r = a \; ; \; U = 0 & \wedge \quad 1CF. \\ r = d \; ; \; U = U_0 & \quad 2CF: \end{cases}$$

Reemplazando en (1):

$$1CF: 0 = \left(A_0 a^0 + \frac{B_0}{a}\right)(1) + \left(A_1 a' + \frac{B_1}{a^2}\right)\cos\theta + \left(A_2 a^2 + \frac{B_2}{a^2}\right)\left[\frac{1}{2}(3\cos^2\theta - 1)\right] + \dots$$

$$0 = A_0 + \frac{B_0}{a} \dots (2) \quad ; \quad A_1 a + \frac{B_1}{a^2} = 0 \dots (3) \quad A_m = 0 \quad m \geq 2$$
$$B_m = 0 \quad m \geq 2$$

$$2^{da}\ C.F: \quad U_0 = A_0 + \frac{B_0}{d} + \left(A_1 d + \frac{B_1}{d^2}\right)\cos\theta + \left(A_2 d^2 + \frac{B_2}{d^3}\right)\left[\frac{1}{2}(3\cos^2\theta - 1)\right] + \dots$$

$$U_0 = A_0 + \frac{B_0}{d} \dots (4) \quad ; \quad A_1 d + \frac{B_1}{d^2} = 0 \dots (5)$$

Relacionando: (2) y (4) $\longleftrightarrow$ (3) y (5)

$$\begin{cases} 0 = A_0 + \frac{B_0}{a} \\ U_0 = A_0 + \frac{B_0}{d} \end{cases} (-) \wedge \begin{cases} A_1 a + \frac{B_1}{a^2} = 0 \\ A_1 d + \frac{B_1}{d^2} = 0 \end{cases} (-) \Rightarrow$$ Resolviendo sistemas de ecuaciones.

$$U_0 = B_0\left(\frac{1}{d} - \frac{1}{a}\right) \quad \wedge \quad A_1(d-a) = B_1\left(\frac{1}{d^2} - \frac{1}{d^2}\right) = B_1\left(\frac{1}{a^2} - \frac{1}{d^2}\right)$$

$$\Rightarrow \boxed{B_0 = \frac{U_0 a d}{a-d}} \qquad \Rightarrow \frac{B_1}{A_1} = \frac{(d-d)}{\left(\frac{1}{d^2} - \frac{1}{d^2}\right)} \Rightarrow \boxed{\begin{aligned} B_1 &= d-a \\ A_1 &= \frac{1}{d^2} - \frac{1}{d^2} \end{aligned}}$$

$$\boxed{A_0 = -\frac{B_0}{a} = \frac{U_0 a d}{a(d-a)}}$$

Reemplazando los coeficientes en la solución general (1):

$$U(r,\theta) = A_0 r^0 + B_0 r^{-1} + (A_1 r' + B_1 r^{-2})\cos\theta$$

$$U(r,\theta) = \frac{U_0 a d}{a(d-a)} + \frac{U_0 a d}{(a-d)}\frac{1}{r} + \left[\left(\frac{1}{a^2 d^2}\right)r + \left(\frac{d-d}{r^2}\right)\right]\cos\theta$$

$$\therefore U(r,\theta) = \frac{U_0}{d-a} + \left(\frac{a d . U_0}{a-d}\right)\frac{1}{r} + \left(\frac{d^2 - a^2}{a^2 d^2}\right) r\cos\theta + \left(\frac{d-a}{r^2}\right)\cos\theta$$

Condición:

Para: $\theta = 0°$ $\quad r = d/2$

$$U(r,\theta) = \frac{U_0}{d-a} + \frac{2a\, U_0}{a-d} + \left(\frac{d^2 - a^2}{a^2 d^2}\right)\frac{d}{2}\cos\theta + \frac{(d-a)}{\frac{d^2}{4}}\cos\theta$$

$$U(d/2, 0) = U(r,\theta) = \frac{U_0}{d-a} + \frac{2a\, U_0}{a-d} + \frac{1}{2}\left(\frac{d^2 - a^2}{a^2 d}\right) + \frac{4(d-a)}{d^2}$$

Hallando el campo eléctrico y sus condiciones:

$$\bar{E} = -\nabla U$$

$$\bar{E} = -\left[\frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta}\right]$$

$$\frac{dU}{dr} = \left(-\frac{U_0 a d}{(a-d)r^2} + \left(\frac{d^2-a^2}{a^2d^2}\right)\cos\theta - \frac{2(d-a)\cos\theta}{r^3}\right)\hat{r}$$

$$\frac{dU}{d\theta} = \cancel{-\left[\left(\frac{d^2-a^2}{a^2d^2}\right)r - \left(\frac{d-a}{r^2}\right)\right]\operatorname{sen}\theta} = \left[-r\frac{(d^2-a^2)\operatorname{sen}\theta}{a^2d^2} - \left(\frac{d-a}{r^2}\right)\operatorname{sen}\theta\right]\hat{\theta}$$

$$\Rightarrow \bar{E} = -\Big[$$

$$\bar{E} = \left[\frac{U_0 a d}{(a-d)r^2} - \left(\frac{d^2-a^2}{a^2d^2}\right)\cos\theta + \frac{2(d-a)\cos\theta}{r^3}\right]\hat{r} + \left[\frac{r(d^2-a^2)}{a^2d^2}\operatorname{sen}\theta + \left(\frac{d-a}{r^2}\right)\operatorname{sen}\theta\right]\hat{\theta}$$

$$\bar{E}(d/2; 0°) = \left[\frac{4U_0 a d}{(a-d)d^2} - \left(\frac{d^2-a^2}{a^2d^2}\right) + \frac{16(d-a)}{d^2}\right]\hat{r} + \left[\cancel{\frac{d(d^2-a^2)}{2a^2d^2}(0)} + \cancel{4\left(\frac{d-a}{d^2}\right)(0)}\right]\hat{\theta}$$

$$\therefore \bar{E}(d/2, 0°) = \left[\frac{4U_0 a d}{(a-d)d^2} - \left(\frac{d^2-a^2}{a^2d^2}\right) + \frac{16(d-a)}{d^2}\right]\hat{r}$$